

# GLOBAL REGULARITY FOR SUPERCRITICAL NONLINEAR DISSIPATIVE WAVE EQUATIONS IN 3D

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**ABSTRACT.** The nonlinear wave equation  $u_{tt} - \Delta u + |u_t|^{p-1}u_t = 0$  is shown to be globally well-posed in the Sobolev spaces of radially symmetric functions  $H_{\text{rad}}^k(\mathbb{R}^3) \times H_{\text{rad}}^{k-1}(\mathbb{R}^3)$  for all  $p \geq 3$  and  $k \geq 3$ . Moreover, global  $C^\infty$  solutions are obtained when the initial data are  $C_0^\infty$  and exponent  $p$  is an odd integer.

The radial symmetry allows a reduction to the one-dimensional case where an important observation of A. Haraux [6] can be applied, i.e., dissipative nonlinear wave equations contract initial data in  $W^{k,q}(\mathbb{R}) \times W^{k-1,q}(\mathbb{R})$  for all  $k \in [1, 2]$  and  $q \in [1, \infty]$ .

## 1. INTRODUCTION

Dissipative nonlinear wave equations are well-posed in  $H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$  for all  $k \in [1, 2]$  and  $n \geq 1$  under the monotonicity condition of Lions and Strauss [14]. This global result makes no exception for nonlinear dissipations of supercritical power, as determined by invariant scaling, and gets around the stringent conditions for well-posedness of general nonlinear wave equations; see Ponce and Sideris [16], Lindblad [13], Wang and Fang [25] and the references therein.

The monotonicity method has been less effective in studying higher regularity. It is still an open question whether supercritical problems are globally well-posed in Sobolev spaces with index  $k > 2$ . The purpose of this paper is to give an affirmative answer when  $n = 3$  and initial data have radial symmetry. To state the result, let  $\square u = u_{tt} - \Delta u$  be the d'Alembertian in  $\mathbb{R}^{3+1}$  and  $Du = (\nabla u, \partial_t u)$  be the space-time gradient of  $u$ . Standard notations are also  $\|\cdot\|_q$ , for the norm in  $L^q(\mathbb{R}^3)$  with  $q \in [1, \infty]$ , and  $D^\alpha$ , for the partial derivative of integer order  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ .

We consider the dissipative nonlinear wave equation

$$(1.1) \quad \square u + |u_t|^{p-1}u_t = 0, \quad x \in \mathbb{R}^3, \quad t > 0,$$

with the Cauchy data

$$(1.2) \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad x \in \mathbb{R}^3.$$

Our main assumptions are  $p \geq 3$  and  $(u_0, u_1) \in H_{\text{rad}}^k(\mathbb{R}^3) \times H_{\text{rad}}^{k-1}(\mathbb{R}^3)$ , where the radially symmetric Sobolev spaces are defined as

$$H_{\text{rad}}^k(\mathbb{R}^3) = \{u \in H^k(\mathbb{R}^3) : (x_i \partial_{x_j} - x_j \partial_{x_i})u(x) = 0 \text{ for } 1 \leq i < j \leq 3\}.$$

Clearly, such  $u(x)$  depend only on  $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ . The radially symmetric spaces are invariant under the evolution determined by (1.1), (1.2); see [20], [19].

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**Theorem 1.1.** *Assume that  $p \geq 3$  and  $(u_0, u_1) \in H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$ . Then problem (1.1), (1.2) admits a unique global solution  $u$ , such that*

$$D^\alpha u \in C([0, \infty), H_{\text{rad}}^{3-|\alpha|}(\mathbb{R}^3)), \quad |\alpha| \leq 3.$$

**Corollary 1.2.** *The global solution of problem (1.1), (1.2), given by Theorem 1.1, satisfies the following uniform estimates: for  $t \geq 0$ ,*

$$\begin{aligned} \sum_{1 \leq |\alpha| \leq 2} \int_0^t \|D^\alpha u(s)\|_\infty^2 ds &\leq C_1(u_0, u_1), \\ \sum_{|\alpha| \leq 2} \|DD^\alpha u(t)\|_2 &\leq C_2(u_0, u_1), \end{aligned}$$

where  $C_j(u_0, u_1)$ ,  $j = 1, 2$ , are finite whenever  $\|u_0\|_{H^3} + \|u_1\|_{H^2}$  is finite.

It is easy to derive the propagation of higher regularity from Theorem 1.1 and Sobolev embedding inequalities, if the nonlinearity allows further differentiation.

**Theorem 1.3.** *Let  $p$  be an odd integer and  $(u_0, u_1) \in H_{\text{rad}}^k(\mathbb{R}^3) \times H_{\text{rad}}^{k-1}(\mathbb{R}^3)$  with an integer  $k \geq 4$ . Problem (1.1), (1.2) admits a unique global solution  $u$ , such that*

$$D^\alpha u \in C([0, \infty), H_{\text{rad}}^{k-|\alpha|}(\mathbb{R}^3)), \quad |\alpha| \leq k.$$

In addition, the following estimates hold uniformly in  $t \geq 0$ :

$$\begin{aligned} \sum_{1 \leq |\alpha| \leq k-1} \int_0^t \|D^\alpha u(s)\|_\infty^2 ds &\leq C_1^{(k)}(u_0, u_1), \\ \sum_{|\alpha| \leq k-1} \|DD^\alpha u(t)\|_2 &\leq C_2^{(k)}(u_0, u_1), \end{aligned}$$

where  $C_j^{(k)}(u_0, u_1)$ ,  $j = 1, 2$ , are finite whenever  $\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}$  is finite.

If  $(u_0, u_1) \in C_{\text{rad}}^\infty(\mathbb{R}^3) \times C_{\text{rad}}^\infty(\mathbb{R}^3)$  and  $(u_0, u_1)$  are compactly supported, then problem (1.1), (1.2) admits a unique global solution

$$u \in C^\infty([0, \infty), C_{\text{rad}}^\infty(\mathbb{R}^3)),$$

such that  $u(\cdot, t)$  is also compactly support for all  $t \geq 0$ .

The above results rely heavily on the dissipative nonlinearity and radial symmetry. In less favorable circumstances, global regularity is well beyond the reach of existing methods which require the nonlinearity be slightly weaker than the linear differential operator. Nonlinear conservative wave equations and Schrodinger equations, for example, are well-understood only in subcritical and critical cases; see Struwe [22], Grillakis [4], Shatah and Struwe [18], Nakanishi [15] for the former and Bourgain [1], Grillakis [5], Colliander, Keel, Staffilani, Takaoka and Tao [3] for the latter. The supercritical nonlinearities allow mostly conditional results about the equivalence of certain norms and nature of blow up, such as Kenig and Merle [9] and Killip and Visan [10]. Exceptions are the well-posedness for log-supercritical nonlinearities and radial data by Tao [23] and the well-posedness and scattering for log-log-supercritical nonlinearities by Roy [17].

Dissipative and conservative nonlinear wave equations behave very differently even in Sobolev spaces with  $k \leq 2$  derivatives. This is evident from the invariant scaling of (1.1):  $u(x, t) \mapsto u_\lambda(x, t) = \lambda^{(2-p)/(p-1)} u(\lambda x, \lambda t)$ , with  $\lambda > 0$ . Then

$$(1.3) \quad \|u_\lambda(t)\|_{\dot{H}^k} = \lambda^{(2-p)/(p-1)+k-3/2} \|u(\lambda t)\|_{\dot{H}^k},$$

so the critical space for well-posedness is  $\dot{H}^{k_c}(\mathbb{R}^3) \times \dot{H}^{k_c-1}(\mathbb{R}^3)$  with

$$k_c = 3/2 + (p-2)/(p-1).$$

However, the result of [14] shows that  $k \in [1, 2]$  is sufficient for any  $p > 1$ . Here the monotone dissipation plays a decisive role, since (1.1) remains dissipative after applying  $D^\alpha$  with  $|\alpha| = 1$ :

$$(1.4) \quad \square D^\alpha u + p|u_t|^{p-1} D^\alpha u_t = 0.$$

We can use  $u_t$  and  $D^\alpha u_t$  as multipliers for (1.1) and (1.4), respectively, to obtain

$$\begin{aligned} \|Du(t)\|_2^2 + 2 \int_0^t \|u_s(s)\|_{p+1}^{p+1} ds &= \|Du(0)\|_2^2, \\ \|DD^\alpha u(t)\|_2^2 + 2p \int_0^t \| |u_s(s)|^{(p-1)/2} D^\alpha u_s(s) \|_2^2 ds &= \|DD^\alpha u(0)\|_2^2. \end{aligned}$$

As derivatives of order  $k \leq 2$  turn out to be *a priori* bounded, the corresponding well-posedness results readily follow from the monotonicity method of [14].

Two differentiations of (1.1) yield an equation that is no longer dissipative. It becomes a nontrivial task to derive uniform estimates for derivatives of order three and higher. Some information is provided by the invariant scaling, which predicts the non-concentration of second-order norms if  $k = 2$  and  $(2-p)/(p-1) + 1/2 > 0$  in (1.3). Thus, the range of subcritical exponents is  $p < 3$ . For the complementary range  $p \geq 3$  and  $k > 2$ , the global well-posedness of (1.1) in  $H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$  is a difficult problem. Until recently, the proof has been known only in the critical case  $p = 3$  with radially symmetric data [24].

This paper shows that no critical exponent exists for the regularity of dissipative nonlinear wave equations with radial symmetry. The development of singularities is prevented by the monotonicity of seminorms involving second-order derivatives. We actually obtain, after setting  $r = |x|$  and  $X_\pm(r, t) = (\partial_t \pm \partial_r)(r\partial_t u)$ , that

$$\max_{\pm} \sup_{r>0} |X_\pm(r, t)| \leq \max_{\pm} \sup_{r>0} |X_\pm(r, 0)|, \quad t \geq 0.$$

Such decreasing quantities exist only for linear and dissipative nonlinear wave equations in  $\mathbb{R}^3$  with radial data. The proof is based on differentiating (1.1) in  $t$  and rewriting the equation for  $\partial_t u$  as

$$(\partial_t \mp \partial_r)X_\pm + \frac{p}{2}|\partial_t u|^{p-1}(X_+ + X_-) = 0.$$

A. Haraux [6] has applied the same idea to derive  $W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$  estimates in the one-dimensional case. In higher dimensions with radial symmetry, similar  $W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})$  estimates away from  $r = 0$  are used by Joly, Metivier and Rauch [8], Liang [12] and Carles and Rauch [2]. Since the one-dimensional reduction does not work for general data, the global well-posedness in  $H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ , with  $k > 2$ , remains completely open.

The rest of this paper is organized as follows. Section 2 contains several basic facts and estimates for the wave equation with radially symmetric data in  $\mathbb{R}^3$ . The regularity problem in  $H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$  is studied in Section 3. In the final Section 4, we establish Theorem 1.3 about well-posedness in  $H_{\text{rad}}^k(\mathbb{R}^3) \times H_{\text{rad}}^{k-1}(\mathbb{R}^3)$  with  $k \geq 4$  and  $C_{\text{rad}}^\infty(\mathbb{R}^3) \times C_{\text{rad}}^\infty(\mathbb{R}^3)$  with compact support.

## 2. BASIC ESTIMATES

First of all, we state the energy estimates and Strichartz estimates for the radial wave equation in  $\mathbb{R}^3 \times \mathbb{R}$ .

**Lemma 2.1.** *Let  $u$  be a solution of the Cauchy problem in  $\mathbb{R}^3 \times \mathbb{R}$*

$$\square u = F, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1.$$

(a) *For any source and initial data,  $u$  satisfies the energy estimate*

$$\|Du(t)\|_2 \leq C(\|\nabla u_0\|_2 + \|u_1\|_2) + C \int_0^t \|F(s)\|_2 ds$$

*with an absolute constant  $C$  for all  $t \geq 0$ .*

(b) *For radial source and initial data,  $u$  is also a radial function which satisfies*

$$\left( \int_0^t \|u(s)\|_\infty^2 ds \right)^{1/2} \leq C(\|\nabla u_0\|_2 + \|u_1\|_2) + C \int_0^t \|F(s)\|_2 ds$$

*for all  $t \geq 0$ .*

Part (a) can be found in Strauss [20], Hörmander [7], and Shatah and Struwe [19]. Estimate (b) is the so-called “radial Strichartz estimate” in 3D. Klainerman and Machedon [11] have found the homogeneous version of (b) which implies the non-homogeneous estimate stated here.

The following is a collection of useful facts concerning local solvability and other properties of problem (1.1), (1.2). These results can be found in [20], [7] and [19].

**Lemma 2.2.** *Let  $k \geq 3$  and  $(u_0, u_1) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ .*

(a) *There exists  $T > 0$ , such that problem (1.1), (1.2) has a unique solution  $u$  satisfying*

$$D^\alpha u \in C([0, T], H^{k-|\alpha|}(\mathbb{R}^3)), \quad |\alpha| \leq k.$$

*Moreover, we have*

$$\sup_{t \in [0, T]} \|D^\alpha u(t)\|_2 \leq C_k,$$

*where  $T$  and  $C_k$  can be chosen to depend continuously on  $\|u_0\|_{H^k} + \|u_1\|_{H^{k-1}}$ .*

(b) *The continuation principle holds: if  $T_* = T_*(u_0, u_1)$  is the supremum of all numbers  $T$  for which (a) holds, then either  $T_* = \infty$  or*

$$\sup_{t \in [0, T_*)} \|D^\alpha u(t)\|_2 = \infty$$

*for some  $\alpha$  with  $|\alpha| \leq k$ .*

(c) *If the data  $(u_0, u_1)$  are radially symmetric, the solution  $(u, u_t)$  is also radially symmetric.*

Next, we state two preliminary estimates for problem (1.1), (1.2). These results, called the energy dissipation laws, are already discussed in the introduction.

**Lemma 2.3.** *Assume that  $(u_0, u_1) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$  and  $|\alpha| = 1$ . Let  $u$  be the solution of problem (1.1), (1.2) for  $t \in [0, T]$ , given by Lemma 2.2. Then*

$$\begin{aligned} \frac{1}{2} \|Du(t)\|_2^2 + \int_0^t \|u_s(s)\|_{p+1}^{p+1} ds &= \frac{1}{2} \|Du(0)\|_2^2, \\ \frac{1}{2} \|DD^\alpha u(t)\|_2^2 + p \int_0^t \| |u_s(s)|^{(p-1)/2} D^\alpha u_s(s) \|_2^2 ds &= \frac{1}{2} \|DD^\alpha u(0)\|_2^2, \end{aligned}$$

for all  $t \in [0, T]$ . Thus, the following norms of  $u$  are uniformly bounded:

$$\begin{aligned} \|Du(t)\|_2 &\leq \|Du(0)\|_2, \quad \|DD^\alpha u(t)\|_2 \leq \|DD^\alpha u(0)\|_2, \quad |\alpha| \leq 1, \\ \int_0^T (\|u_s(s)\|_{p+1}^{p+1} + \| |u_s(s)|^{(p-1)/2} D^\alpha u_s(s) \|_2^2) ds &\leq \|Du(0)\|_2^2 + \|DD^\alpha u(0)\|_2^2. \end{aligned}$$

**Proof.** Multiplying equation (1.1) by  $u_t$ , we get

$$0 = (\square u + |u_t|^{p-1} u_t) u_t = \left( \frac{|Du|^2}{2} \right)_t - \operatorname{div}(u_t \nabla u) + |u_t|^{p+1}.$$

The first-order energy identity follows from the integration on  $\mathbb{R}^3$  and divergence theorem if  $u(x, t)$  has compact support with respect to  $x$ . More generally, we can approximate  $(u_0(x), u_1(x))$  with compactly supported  $C^\infty$  functions and use the finite propagation speed to show that the boundary integral of  $\operatorname{div}(u_t \nabla u)$  is zero. Property (a) in Lemma 2.2 implies that the approximations will converge to the actual solution.

Similarly, we can differentiate equation (1.1) and multiply with  $D^\alpha u_t(x, t)$  to show the second-order energy identity. Let us recall that we work with solutions whose third-order derivatives belong to  $L^2(\mathbb{R}^3)$ .  $\square$

Finally, we state the strong version of Strauss inequality. The original version can be found as Radial Lemma 1 of Strauss [21].

**Lemma 2.4.** (*Strong version of Strauss Inequality* [21]) *Let  $U \in H_{\text{rad}}^1(\mathbb{R}^3)$ . There exists a constant  $C > 0$ , such that for every  $R > 0$*

$$R|U(R)| \leq C\|U\|_{H^1(|x|>R)} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

### 3. GLOBAL EXISTENCE OF RADIAL SOLUTIONS IN $H^3 \times H^2$

The following lemma is essential for the proof of Theorem 1.1. This approach to  $L^\infty$  estimates of second-order derivatives is borrowed from Haraux [6].

**Lemma 3.1.** *Let  $(u_0, u_1) \in H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$  and let  $u$  be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2. There exists a positive constant  $C = C(\|u_0\|_{H^3}, \|u_1\|_{H^2})$ , such that*

$$(3.1) \quad \sup_{r \in [0, \infty)} r |\partial_t^2 u(r, t)| \leq C,$$

$$(3.2) \quad \sup_{r \in [0, \infty)} |\partial_t u(r, t)| \leq C, \quad \sup_{r \in [0, \infty)} r |\partial_r \partial_t u(r, t)| \leq C,$$

for all  $t \in [0, T_*)$ .

**Proof.** Since the solution of (1.1) is radially symmetric, the equation becomes

$$(3.3) \quad (\partial_t^2 - \partial_r^2)(ru) + r|\partial_t u|^{p-1} \partial_t u = 0,$$

where  $r = |x|$ . Let us define

$$X_\pm(r, t) = (\partial_t \pm \partial_r)(r \partial_t u).$$

We differentiate (3.3) in  $t$  to obtain an equation for  $\partial_t u$ . Then we multiply through by  $X_\pm^{2k-1}$  with  $k \in \mathbb{N}$  to obtain

$$\frac{(\partial_t - \partial_r)}{2k} \{X_+(r, t)\}^{2k} + pr |\partial_t u|^{p-1} \partial_t^2 u \{X_+(r, t)\}^{2k-1} = 0$$

and

$$\frac{(\partial_t + \partial_r)}{2k} \{X_-(r, t)\}^{2k} + pr |\partial_t u|^{p-1} \partial_t^2 u \{X_-(r, t)\}^{2k-1} = 0.$$

Adding the above two equations and using  $X_+(r, t) + X_-(r, t) = 2r \partial_t^2 u$ , we have

$$\frac{(\partial_t - \partial_r)}{2k} X_+^{2k} + \frac{(\partial_t + \partial_r)}{2k} X_-^{2k} + p |\partial_t u|^{p-1} \frac{X_+ + X_-}{2} (X_+^{2k-1} + X_-^{2k-1}) = 0.$$

Here we remark that the following inequality holds for  $a, b \in \mathbf{R}$  and  $k \in \mathbf{N}$ :

$$(a + b)(a^{2k-1} + b^{2k-1}) \geq 0.$$

From this inequality with  $a = X_+$  and  $b = X_-$ , we derive

$$\partial_t (X_+^{2k} + X_-^{2k}) + \partial_r (X_-^{2k} - X_+^{2k}) \leq 0.$$

It is easy to see that integration on  $[0, R] \times [0, t]$  implies

$$\begin{aligned} \int_0^R \{X_+^{2k}(r, t) + X_-^{2k}(r, t)\} dr &\leq \int_0^R \{X_+^{2k}(r, 0) + X_-^{2k}(r, 0)\} dr \\ &\quad + \int_0^t \{X_+^{2k}(R, s) - X_-^{2k}(R, s)\} ds, \end{aligned}$$

since  $X_-^{2k}(0, t) - X_+^{2k}(0, t) = 0$ . We also notice that  $|X_\pm(R, t)| \rightarrow 0$  as  $R \rightarrow \infty$ , which is a consequence of Lemma 2.4. Thus, we get

$$\|X_+(\cdot, t)\|_{L^{2k}(\mathbb{R}_+)}^{2k} + \|X_-(\cdot, t)\|_{L^{2k}(\mathbb{R}_+)}^{2k} \leq \|X_+(\cdot, 0)\|_{L^{2k}(\mathbb{R}_+)}^{2k} + \|X_-(\cdot, 0)\|_{L^{2k}(\mathbb{R}_+)}^{2k}.$$

The two terms on the right hand side satisfy

$$\|X_\pm(\cdot, 0)\|_{L^{2k}(\mathbb{R}_+)}^{2k} \leq \|X_\pm(\cdot, 0)\|_{L^\infty(\mathbb{R}_+)}^{2k-2} \|X_\pm(\cdot, 0)\|_{L^2(\mathbb{R}_+)}^2,$$

where  $\|X_\pm(\cdot, 0)\|_{L^2(\mathbb{R}_+)} < \infty$  by Lemma 2.4 and  $(u_0, u_1) \in H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$ . This observation yields a more convenient estimate:

$$\begin{aligned} \|X_+(\cdot, t)\|_{L^{2k}(\mathbb{R}_+)}^{2k} + \|X_-(\cdot, t)\|_{L^{2k}(\mathbb{R}_+)}^{2k} &\leq \|X_+(\cdot, 0)\|_{L^\infty(\mathbb{R}_+)}^{2k-2} \|X_+(\cdot, 0)\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \|X_-(\cdot, 0)\|_{L^\infty(\mathbb{R}_+)}^{2k-2} \|X_-(\cdot, 0)\|_{L^2(\mathbb{R}_+)}^2. \end{aligned}$$

Letting  $k \rightarrow \infty$ , we have that

$$\begin{aligned} &\max\{\|X_+(\cdot, t)\|_{L^\infty(\mathbb{R}_+)}, \|X_-(\cdot, t)\|_{L^\infty(\mathbb{R}_+)}\} \\ (3.4) \quad &\leq \max\{\|X_+(\cdot, 0)\|_{L^\infty(\mathbb{R}_+)}, \|X_-(\cdot, 0)\|_{L^\infty(\mathbb{R}_+)}\}. \end{aligned}$$

The remaining part of the proof is standard. Since  $X_+(r, t) + X_-(r, t) = 2r \partial_t^2 u$ , claim (3.1) follows from (3.4). We can write  $2\partial_r(r \partial_t u) = X_+(r, t) - X_-(r, t)$  and

$$2r |\partial_t u(r, t)| \leq \int_0^r |X_+(\rho, t) - X_-(\rho, t)| d\rho \leq Cr.$$

Hence  $\|\partial_t u(\cdot, t)\|_{L^\infty(\mathbb{R}_+)} \leq C/2 < \infty$ , which is the first claim in (3.2). We finally observe that  $2r \partial_r \partial_t u(r, t) = X_+(r, t) - X_-(r, t) - 2\partial_t u(r, t)$ , so the second claim in (3.2) follows from the first one and (3.4). The proof is complete.  $\square$

**Proof of Theorem 1.1.** It is sufficient to show that  $\|u(t)\|_{H^3} + \|u_t(t)\|_{H^2}$  does not blow up as  $t \rightarrow T_*$ , where  $u$  is the local solution of (1.1) given by Lemma 2.2. The first and second order norms of  $u$  are *a priori* bounded from Lemma 2.3, so the global existence of  $u$  is guaranteed by the next result about third order norms.

**Proposition 3.2.** *Assume that  $(u_0, u_1) \in H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$  and let  $u$  be the local solution of problem (1.1), (1.2) constructed in Lemma 2.2. There exists a positive constant  $C = C(u_0, u_1, T_*)$  such that*

$$\sum_{|\alpha|=3} \|D^\alpha u(t)\|_2 \leq C(u_0, u_1, T_*)$$

for all  $t \in [0, T_*)$ .

**Proof.** Differentiating (1.1) twice, we find that  $D^\alpha u$  is a weak solution of

$$\square D^\alpha u + p|u_t|^{p-1} D^\alpha u_t + p(p-1)|u_t|^{p-3} u_t D^{\alpha_1} u_t D^{\alpha_2} u_t = 0,$$

where  $\alpha = \alpha_1 + \alpha_2$  with  $|\alpha_1| = 1$  and  $|\alpha_2| = 1$ . Thus,  $D^\alpha u$  satisfies the energy estimate in Lemma 2.1 (a):

$$\begin{aligned} \|DD^\alpha u(t)\|_2 &\leq C\|DD^\alpha u(0)\|_2 \\ &\quad + C \int_0^t \|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 ds \\ &\quad + C \int_0^t \|u_s^{p-2}(s) D^{\alpha_1} u_s(s) D^{\alpha_2} u_s(s)\|_2 ds \end{aligned}$$

for  $t \in [0, T_*)$ . We notice that

$$\|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 \leq C \|D^\alpha u_s(s)\|_2,$$

by (3.1) in Lemma 3.1, and

$$\|u_s^{p-2}(s) D^{\alpha_1} u_s(s) D^{\alpha_2} u_s(s)\|_2 \leq C \left\| \frac{D^{\alpha_1} u_s(s)}{|\cdot|} \right\|_2,$$

by (3.1) and (3.2) in Lemma 3.1. Thus,

$$\begin{aligned} \|DD^\alpha u(t)\|_2 &\leq C\|DD^\alpha u(0)\|_2 \\ &\quad + C \int_0^t \|D^\alpha u_s(s)\|_2 ds \\ &\quad + C \int_0^t \left\| \frac{D^{\alpha_1} u_s(s)}{|\cdot|} \right\|_2 ds. \end{aligned}$$

The third term of the right hand can be estimated by Hardy's inequality:

$$\begin{aligned} \|DD^\alpha u(t)\|_2 &\leq C\|DD^\alpha u(0)\|_2 \\ &\quad + C \int_0^t \|D^\alpha u_s(s)\|_2 ds \\ &\quad + C \int_0^t \|DD^{\alpha_1} u_s(s)\|_2 ds. \end{aligned}$$

We add these estimates for all  $|\alpha| = 2$  to get

$$\sum_{|\alpha|=3} \|D^\alpha u(t)\|_2 \leq C \sum_{|\alpha|=3} \|D^\alpha u(0)\|_2 + C \int_0^t \sum_{|\alpha|=3} \|D^\alpha u(s)\|_2 ds.$$

Making use of Gronwall's inequality, we finally have

$$\sum_{|\alpha|=3} \|D^\alpha u(t)\|_2 \leq C \sum_{|\alpha|=3} \|D^\alpha u(0)\|_2 e^{Ct}.$$

This completes the proof of global existence.  $\square$

**Proof of Corollary 1.2.** We can now verify the uniform estimates of  $L^2$  norms and square integrability of  $L^\infty$  norms on  $[0, \infty)$ . Notice that inequality (b) in Lemma 2.1 holds on any interval  $[t_0, t]$  for all  $|\alpha| = 1$ . Hence we get

$$\begin{aligned} \sum_{|\alpha|=1} \left( \int_{t_0}^t \|D^\alpha u(s)\|_\infty^2 ds \right)^{1/2} &\leq C \sum_{|\alpha|=1} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C \sum_{|\alpha|=1} \int_{t_0}^t \|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 ds. \end{aligned}$$

It is convenient to abbreviate the left hand side as

$$N_1(t) = \sum_{|\alpha|=1} \left( \int_{t_0}^t \|D^\alpha u(s)\|_\infty^2 ds \right)^{1/2}, \quad t \geq t_0,$$

and estimate the integrand on the right hand side as

$$\|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 \leq C \|D^\alpha u(s)\|_\infty \|u_s^{(p-1)/2}(s) D^\alpha u_s(s)\|_2,$$

from (3.2). Applying the Cauchy inequality on  $[t_0, t]$  to the initial estimate,

$$\begin{aligned} N_1(t) &\leq C \sum_{|\alpha|=1} \|DD^\alpha u(t_0)\|_2 \\ &\quad + CN_1(t) \left( \int_{t_0}^t \|u_s^{(p-1)/2}(s) Du_s(s)\|_2^2 ds \right)^{1/2}. \end{aligned}$$

The above integral converges on  $[0, \infty)$  by Lemma 2.3. We can find  $t_0$ , such that

$$(3.5) \quad C \left( \int_{t_0}^t \|u_s^{(p-1)/2}(s) Du_s(s)\|_2^2 ds \right)^{1/2} \leq \frac{1}{2}, \quad t \geq t_0.$$

Thus,  $N_1(t) \leq 2C \sum_{|\alpha|=1} \|DD^\alpha u(t_0)\|_2$  for all  $t \geq t_0$ . If  $t < t_0$ , a similar estimate for

$N_1(t)$  follows from Theorem 1.1. Hence  $\|D^\alpha u(\cdot)\|_\infty \in L^2([0, \infty))$  for  $|\alpha| = 1$ .

Next, we show that the second order norms in Corollary 1.2 are also uniformly bounded. Set

$$N_2(t) = \sum_{|\alpha|=2} \left( \int_{t_0}^t \|D^\alpha u(s)\|_\infty^2 ds \right)^{1/2}, \quad t \geq t_0,$$

and apply estimate (b) in Lemma 2.1 to obtain

$$\begin{aligned} \left( \int_{t_0}^t \|D^\alpha u(s)\|_\infty^2 ds \right)^{1/2} &\leq C \|DD^\alpha u(t_0)\|_2 + C \int_{t_0}^t \|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 ds \\ &\quad + C \int_{t_0}^t \|u_s^{p-2}(s) (Du_s(s))^2\|_2 ds, \end{aligned}$$

where  $|\alpha| = 2$ . Noticing that (3.2) gives

$$(3.6) \quad \|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 \leq C \|u_s(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2,$$

$$(3.7) \quad \|u_s^{p-2}(s) (Du_s(s))^2\|_2 \leq C \|D^\alpha u(s)\|_\infty \|u_s^{(p-1)/2}(s) Du_s(s)\|_2$$



and applying the Cauchy inequality on  $[t_0, t]$ , we get

$$\begin{aligned} N_2(t) &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C \int_{t_0}^t \|u_s(s)\|_\infty^2 \left( \sum_{|\alpha|=2} \|D^\alpha u_s(s)\|_2 \right) ds \\ &\quad + CN_2(t) \left( \int_{t_0}^t \|u_s^{(p-1)/2}(s) Du_s(s)\|_2^2 ds \right)^{1/2}. \end{aligned}$$

We use again (3.5) to derive

$$\begin{aligned} N_2(t) &\leq 2C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + 2C \int_{t_0}^t \|u_s(s)\|_\infty^2 \left( \sum_{|\alpha|=2} \|D^\alpha u_s(s)\|_2 \right) ds \end{aligned}$$

for sufficiently large  $t \geq t_0$ . Let us introduce

$$N_3(t) = \sum_{|\alpha|=2} \sup_{s \in [t_0, t]} \|DD^\alpha u(s)\|_2, \quad t \geq t_0.$$

Then we can write

$$(3.8) \quad N_2(t) \leq 2C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 + 2CN_3(t) \left( \int_{t_0}^t \|Du(s)\|_\infty^2 ds \right).$$

We will combine the above estimate with Lemma 2.1 (a) for the interval  $[t_0, t]$ :

$$\begin{aligned} \|DD^\alpha u(t)\|_2 &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C \int_{t_0}^t \|u_s^{p-1}(s) D^\alpha u_s(s)\|_2 ds \\ &\quad + C \int_{t_0}^t \|u_s^{p-2}(s) (Du_s(s))^2\|_2 ds, \end{aligned}$$

where  $t \geq t_0$ . It follows from (3.6) and (3.7) that

$$\begin{aligned} \|DD^\alpha u(t)\|_2 &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + C \int_{t_0}^t \|Du(s)\|_\infty^2 \|D^\alpha u_s(s)\|_2 ds \\ &\quad + C \left( \int_{t_0}^t \|Du_s(s)\|_\infty^2 ds \right)^{1/2} \left( \int_{t_0}^t \|u_s^{(p-1)/2}(s) Du_s(s)\|_2^2 ds \right)^{1/2}. \end{aligned}$$

Summing over  $|\alpha| = 2$  and applying Lemma 2.3, we simplify the estimate to

$$\begin{aligned} N_3(t) &\leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 \\ &\quad + CN_3(t) \left( \int_{t_0}^t \|Du(s)\|_\infty^2 ds \right) \\ &\quad + CN_2(t) \left( \sum_{|\alpha|=1} \|DD^\alpha u(0)\|_2 \right). \end{aligned}$$

Since the integral

$$\int_0^\infty \|Du(s)\|_\infty^2 ds$$

is convergent, we choose a sufficiently large  $t_0$  to finally get

$$(3.9) \quad N_3(t) \leq C \sum_{|\alpha|=2} \|DD^\alpha u(t_0)\|_2 + CN_2(t) \left( \sum_{|\alpha|=1} \|DD^\alpha u(0)\|_2 \right)$$

for  $t \geq t_0$ . Estimates (3.8), (3.9) are sufficient to bound uniformly  $N_2(t) + N_3(t)$ .  $\square$

#### 4. GLOBAL EXISTENCE OF RADIAL SOLUTIONS IN $H^k \times H^{k-1}$ WITH $k > 3$

Here we consider equation (1.1) with a supercritical integer  $p = 2m + 1$ ,  $m \in \mathbf{N}$ . For  $k \geq 3$  and  $t \in [0, T_*)$ , let us define

$$E_k(t) = \sum_{|\alpha|=k-1} \|DD^\alpha u(t)\|_2, \quad N_k(t) = \sum_{1 \leq |\beta| \leq k-1} \left( \int_0^t \|D^\beta u(s)\|_\infty^2 ds \right)^{1/2},$$

where  $u$  is the local solution of (1.1), (1.2) constructed in Lemma 2.2.

**Proof of Theorem 1.3.** We use induction in  $k$ . From Theorem 1.1, we know that the following hold when  $k = 3$ :

$$(4.1) \quad T_* = \infty, \quad \sup_{t \in [0, \infty)} E_k(t) < \infty, \quad \sup_{t \in [0, \infty)} N_k(t) < \infty.$$

Assuming these claims for  $k$ , we will verify them for  $k + 1$ . The proof starts from applying  $D^\alpha$  to (1.1) with  $p = 2m + 1$  and solving for the highest derivatives:

$$\begin{aligned} \square D^\alpha u &= c_{m,1} u_t^{2m} D^\alpha u_t \\ &\quad + c_{m,2} u_t^{2m-1} \sum_{\alpha_1 + \alpha_2 = \alpha} D^{\alpha_1} u_t D^{\alpha_2} u_t \\ &\quad + c_{m,3} u_t^{2m-2} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} D^{\alpha_1} u_t D^{\alpha_2} u_t D^{\alpha_3} u_t \\ &\quad + \dots \\ &\quad + c_{m,l} u_t^{2m+1-l} \sum_{\alpha_1 + \alpha_2 + \dots + \alpha_l = \alpha} D^{\alpha_1} u_t \dots D^{\alpha_l} u_t = 0, \end{aligned} \quad (4.2)$$

where  $l = \min\{2m+1, k\}$ ,  $c_{m,i}$  are constants and  $|\alpha_i| \geq 1$  for  $i = 1, \dots, l$ . Making use of Lemma 2.1 (a), we obtain

$$\begin{aligned}
 \|DD^\alpha u(t)\|_2 &\leq C\|DD^\alpha(0)\|_2 + C \int_0^t \|u_s^{2m} D^\alpha u_s\|_2 ds \\
 &+ \sum_{\alpha_1+\alpha_2=\alpha} C \int_0^t \|u_s^{2m-1} D^{\alpha_1} u_s D^{\alpha_2} u_s\|_2 ds \\
 (4.3) \quad &+ \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} C \int_0^t \|u_s^{2m-2} D^{\alpha_1} u_s D^{\alpha_2} u_s D^{\alpha_3} u_s\|_2 ds \\
 &+ \dots \\
 &+ \sum_{\alpha_1+\alpha_2+\dots+\alpha_l=\alpha} C \int_0^t \|u_s^{2m+1-l} D^{\alpha_1} u_s \dots D^{\alpha_l} u_s\|_2 ds.
 \end{aligned}$$

Only the first integrand involves a derivatives of order  $k+1$ . From (3.2), we get

$$(4.4) \quad \|u_s^{2m} D^\alpha u_s\|_2 \leq C \|u_s\|_\infty^2 E_{k+1}(s).$$

In the second integrand, there are two cases:  $\max\{|\alpha_1|, |\alpha_2|\} = k-1$  and  $\max\{|\alpha_1|, |\alpha_2|\} \leq k-2$ . We rely on (3.2) to derive

$$\begin{aligned}
 \|u_s^{2m-1} D^{\alpha_1} u_s D^{\alpha_2} u_s\|_2 &\leq \|u_s\|_\infty^{2m-2} \|u_s\|_\infty \|Du_s\|_\infty E_k(s) \\
 (4.5) \quad &\leq C \left( \sum_{1 \leq |\beta| \leq 2} \|D^\beta u(s)\|_\infty^2 \right) E_k(s)
 \end{aligned}$$

and

$$\begin{aligned}
 \|u_s^{2m-1} D^{\alpha_1} u_s D^{\alpha_2} u_s\|_2 &\leq \|u_s\|_\infty^{2m-2} \|u_s\|_\infty \|D^\gamma u_s\|_\infty E_{k-|\gamma|+1}(s) \\
 (4.6) \quad &\leq C \left( \sum_{1 \leq |\beta| \leq k-1} \|D^\beta u(s)\|_\infty^2 \right) E_{k-|\gamma|+1}(s),
 \end{aligned}$$

respectively, where  $\gamma$  is such that  $|\gamma| = \max\{|\alpha_1|, |\alpha_2|\}$ .

The third integrand also admits  $\max\{|\alpha_1|, |\alpha_2|, |\alpha_3|\} = k-2$ . Let  $\gamma$  be the multiindex satisfying  $|\gamma| = k-2$ . Applying (3.2), we have

$$\begin{aligned}
 \|u_s^{2m-2} D^{\alpha_1} u_s D^{\alpha_2} u_s D^{\alpha_3} u_s\|_2 &\leq \|u_s\|_\infty^{2m-2} \|D^\gamma u_s\|_\infty \|Du_s\|_\infty E_2(s) \\
 (4.7) \quad &\leq C \left( \sum_{1 \leq |\beta| \leq k-1} \|D^\beta u(s)\|_\infty^2 \right).
 \end{aligned}$$

Finally, we consider all integrands with  $\max\{|\alpha_j| : j = 1, 2, \dots, l\} \leq k-3$ . Since  $k \geq 4$ , we can use the Sobolev embedding and (3.2) to obtain

$$\begin{aligned}
 &\|u_s^{2m+1-l} D^{\alpha_1} u_s D^{\alpha_2} u_s \dots D^{\alpha_l} u_s\|_2 \\
 (4.8) \quad &\leq \|u_s\|_\infty^{2m+1-l} \left( \sum_{1 \leq |\beta| \leq k-2} \|D^\beta u(s)\|_\infty^2 \right) \left( \sum_{j=1}^l E_j(s) \right)^{l-2} \\
 &\leq C \left( \sum_{1 \leq |\beta| \leq k-2} \|D^\beta u(s)\|_\infty^2 \right).
 \end{aligned}$$

We combine the basic estimate (4.3) with estimates (4.4)–(4.8). This yields

$$\|DD^\alpha u(t)\|_2 \leq C\|DD^\alpha u(0)\|_2 + C \int_0^t \|u_s(s)\|_\infty^2 E_{k+1}(s) ds + CN_k(t) + C_k$$

for all  $t \in [0, T_*)$ , where the constant  $C_k = C_k(\|u_0\|_{H^k}, \|u_1\|_{H^{k-1}})$ . Adding all such estimates with  $|\alpha| = k$  and using (4.1), we have

$$E_{k+1}(t) \leq C_{k+1} + C \int_0^t \|u_s(s)\|_\infty^2 E_{k+1}(s) ds,$$

for some constant  $C_{k+1} = C_{k+1}(\|u_0\|_{H^{k+1}}, \|u_1\|_{H^k})$ . From the Gronwall inequality,

$$E_{k+1}(t) \leq C_{k+1} \exp\left(\int_0^t \|u_s(s)\|_\infty^2 ds\right).$$

This shows that  $E_{k+1}(t)$  is uniformly bounded on every finite subinterval of  $[0, T_*)$ . Hence,  $u$  can be continued to a global solution, such that  $\sup_{t \in [0, \infty)} E_{k+1}(t) < \infty$ .

It remains to check the uniform estimates in  $L_t^2 L_x^\infty$ . We apply Lemma 2.1 (b) to the solution of (4.2). The calculations are very similar to the above ones, so we give only the final estimate:

$$\sum_{|\alpha|=k} \left( \int_0^t \|D^\alpha u(s)\|_\infty^2 ds \right)^{1/2} \leq C \int_0^t \|u_s(s)\|_\infty^2 E_{k+1}(s) ds + C_{k+1}.$$

Noticing that  $\sup_{t \in [0, \infty)} E_{k+1}(t) < \infty$  and  $\sup_{t \in [0, \infty)} N_1(t) < \infty$ , we obtain the final estimate in (4.1) with  $k+1$ , i.e.,  $\sup_{t \in [0, \infty)} N_{k+1}(t) < \infty$ . The proof of higher regularity by induction is complete.

Similarly, we can show that  $C^\infty$  regularity is preserved during the evolution of compactly supported radial data.  $\square$

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